4.5. Undetermined Coefficients – Annihilator Approach.

We will study the solution of non-homogeneous equation of constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

$\Longleftrightarrow$

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = g(x)$$

$\Longleftrightarrow$ $Ly = g(x)$,

where $L = a_n D^n + \cdots + a_0$.

FACTORING OPERATOR.

If r_1 is a root of the auxiliary eq:

$$a_n m^n + \cdots + a_1 m + a_0 = 0$$

then

$L = (D - r_1) P(D)$, where $P(D)$ is a polynomial operator of degree $n-1$.

For example,

$$D^2 + 5D + 6 = (D+2)(D+3)$$
$$= (D+3)(D+2)$$

In general, factors of a linear differential operator with constant coefficient is commute.

For example the eq : $y'' + 10y' + 25y = 0$
canbe written as
$$(D^2 + 10D + 25)y = 0$$
or
$$(D+5)^2 y = 0.$$

ANNIHILATOR OPERATOR

Defn. If L is a linear differential operator with constant coeff and f is a sufficiently differentiable function such that
$$L(f(x)) = 0$$
then L is said to be an **annihilator** of the function.

For example: $\boxed{\oplus\ D^n \text{ annihilates each of the functions } 1, x, x^2, \dots, x^{n-1}.}$

That is $D^n x^K = 0 \quad \forall\ 0 \leq K \leq n-1.$

$\boxed{\oplus \text{ The operator } (D-\alpha)^n \text{ annihilates each of the functions } e^{\alpha x}, x e^{\alpha x}, \dots, x^{n-1} e^{\alpha x}.}$

That is $(D-\alpha)^n x^K e^{\alpha x} = 0 \quad \forall\ 0 \leq K \leq n-1.$

Indeed,
$$(D-\alpha)\, x^K e^{\alpha x} = Kx^{K-1} e^{\alpha x} + \alpha x^K e^{\alpha x} - \alpha x^K e^{\alpha x}$$
$$= K x^{K-1} e^{\alpha x}$$
$$(D-\alpha)^2 x^K e^{\alpha x} = K(K-1)\, x^{K-2} e^{\alpha x}$$
$$\Rightarrow (D-\alpha)^K x^K e^{\alpha x} = (K!)\, e^{\alpha x}, \quad (D-\alpha)^{K+1} x^K e^{\alpha x} = 0.$$

Example. Find a differential operator that annihilates the given function.

a) $1-5x^2+8x^3$ b) e^{-3x} c) $4e^{-2x}-10xe^{2x}$.

Solution:

a) D^4 annihilates 1, $-5x^2$, $8x^3$ $\Rightarrow$

$$D^4(1-5x^2+8x^3)=0$$

b) $(D+3)e^{-3x}=0$

c) $(D-2)^2(4e^{-2x}-10xe^{2x})=0$. $\square$

When α, β $(\beta>0)$ are real numbers, the quadratic formula implies that

$$[m^2-2\alpha m+(\alpha^2+\beta^2)]^n=0$$

$$\Leftrightarrow \quad (m-(\alpha+\beta i))^n(m-(\alpha-\beta i))^n=0$$

has complex roots

$\alpha+\beta i$ and $\alpha-\beta i$ both of multiplicity n. From the discussion at the end of section 4.3, we have the following fact.

FACT: The differential operator

$$[D^2+2\alpha D+(\alpha^2+\beta^2)]^n$$

annihilates each of the following functions

$$e^{\alpha x}\cos\beta x,\ e^{\alpha x}x\cos\beta x,\ \dots,\ e^{\alpha x}x^{n-1}\cos\beta x$$

$$e^{\alpha x}\sin\beta x,\ e^{\alpha x}x\sin\beta x,\ \dots,\ e^{\alpha x}x^{n-1}\sin\beta x$$

Example: Find a differential operator that annihilates

$$5e^{-x}\cos 2x - 9e^{-x}\sin 2x$$

Solution:

$$\alpha = -1, \quad \beta = 2$$

$\Rightarrow$ $D^2 + 2D + 5$ annihilates each of the functions $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$. $\square$

When $\alpha = 0$ and $n = 1$ in the previous fact, we have

FACT: $D^2 + \beta^2$ annihilates $\cos \beta x$ and $\sin \beta x$

Properties:

a) If L annihilates y_1 and y_2, then it annihilates any linear combination

$$c_1 y_1 + c_2 y_2$$

b) If L_1 annihilates y_1 and L_2 annihilates y_2, then $L_1 L_2$ $(= L_2 L_1)$ annihilates

$$c_1 y_1 + c_2 y_2.$$

Indeed, $L_1 L_2 (c_1 y_1 + c_2 y_2)$

$$= c_1 L_1 L_2 (y_1) + c_2 L_1 L_2 y_2$$

$$= c_1 L_2 (L_1 (y_1)) + c_2 L_1 (L_2 (y_2))$$

$$= 0.$$

Example: D^2 annihilates $x+5$

D^2+25 annihilates $\cos 5x$

$\Rightarrow$ $D^2(D^2+25)$ annihilates $(x+5+\cos 5x)$.

UNDETERMINED COEFFICIENTS METHOD

Consider the nonhomogeneous linear equation

$$Ly = g(x).$$

Assume that $g(x)$ is a linear combination of functions:

$$K, \quad x^m, \quad x^m e^{\alpha x}, \quad x^m e^{\alpha x}\cos\beta x \text{ and } x^m e^{\alpha x}\sin\beta x.$$

Then we can find an operator L_1 (of the lowest order) that is the product of factor D^n, $(D-\alpha)^n$, $(D^2-2\alpha D+(\alpha^2+\beta^2))^n$.

Applying L_1 to both side of the non-homogeneous eq:

$$L_1 L y = L_1(g(x)) = 0$$

Thus we get a homogeneous eq with higher-order. The solution of eq:

$L_1 L y = 0$ has form

$$y = y_c + y_p$$

where y_c is a solution of Ly and y_p "with appropriate coefficients" will be a particular solution of $Ly = g(x)$.

Example: Solve $y'' + 3y' + 2y = 4x^2$. (2)

Step 1: Solve the homogeneous eq:

$$y'' + 3y' + 2y \quad \text{to get } y_c.$$

The auxiliary eq:

$$m^2 + 3m + 2 = 0 \text{ has solutions}$$

$m_1 = -1$ and $m_2 = -2$.

$$\Rightarrow y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

Step 2: $L_1 = D^3$ annihilates $g(x) = 4x^2$

Thus

$$D^3 (D^2 + 3D + 2) y = D^3 4x^2$$

$$\Leftrightarrow \quad D^3 (D^2 + 3D + 2) y = 0$$

The auxilrary eq of the new eq is

$$m^3 (m+1)(m+2) = 0$$

has $m_1 = m_2 = m_3 = 0$, $m_4 = -1$, $m_4 = -2$

$\Rightarrow$

$$\text{Solution } y = \underbrace{A + Bx + Cx^2}_{y_p} + \underbrace{c_1 e^{-x} + c_2 e^{-2x}}_{y_c}$$

We will find A, B, C s.t. $y_p = Ax + Bx + Cx^2$ is a particular solution of

$$y'' + 3y' + 2y = 4x^2.$$

We have $y_p' = B + 2Cx$, $y_p'' = 2C$

$$\Rightarrow y_p'' + 3y_p' + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2.$$

Comparing coefficients

$$\Leftrightarrow \begin{cases} 2C = 4 \\ 2B + 6C = 0 \\ 2A + 2B + 2C = 0 \end{cases} \Rightarrow \begin{cases} A = 7 \\ B = -6 \\ C = 2 \end{cases}$$

Thus $y_p = 7 - 6x + 2x^2$ is a particular solution.

Step 3: The general solution of eq is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^{-x} + C_2 e^{-2x} + 7 - 6x + 2x^2. \end{aligned}$$

$\square$

Example: Find a particular solution of (may ignore)

$$y'' - y' + y = 2 \sin 3x.$$

Solution:

The natural first guess is some function of form $A \sin 3x$. But when differentiating $\sin 3x$ we get also $\cos 3x$, so the better guess is

$$y_p = A \cos 3x + B \sin 3x.$$

$$\Rightarrow \quad y_p' = -3A \sin 3x + 3B \cos 3x$$

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

We would like to find A, B so that

$$y_p'' - y_p' + y = 2 \sin 3x$$

$$\Leftrightarrow \left(-9A \cos 3x - 9B \sin 3x\right) + 3A \sin 3x + 3 \cos 3x + A \cos 3x + B \sin 3x = 2 \sin 3x$$

$$\Leftrightarrow (-8A - 3B) \cos 3x + (3A - 8B) \sin 3x = 2 \sin 3x$$

or $\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases}$

$\Leftrightarrow \quad A = -\frac{6}{73} \quad , B = -\frac{16}{73}.$

Thus we have a particular solution

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x.$$

Example: (may ignore)

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{3x}$

Solution:

Step 1. Solve the corresponding homogeneous equation:

$$y'' - 2y' - 3y = 0$$

the auxiliary eq:

$$m^2 - 2m - 3 = 0$$

$$\Leftrightarrow (m+1)(m-3) = 0$$

$m_1 = -1, \; m_2 = 3,$

$\Rightarrow \quad y_c = c_1 e^{-x} + c_2 e^{3x}.$

Step 2: Find y_p.

By the superposition principle $y_p = y_{p_1} + y_{p_2}$ where y_{p_1} is a particular solution of

$$y'' - 2y' - 3y = 4x - 5,$$

and y_{p_2} is a particular solution of

$$y'' - 2y' - 3y = 6xe^{2x}.$$

This suggests y_{p_1} has form $Ax+B$, and y_{p_2} has some form $Cxe^{2x} + Ee^{2x}$.

$\Rightarrow$ We guess $y_p = Ax + B + Cxe^{2x} + Ee^{2x}$

$$y_p' = A + 2Cxe^{2x} + Ce^{2x} + 2Ee^{2x}$$
$$y_p'' = 4Cxe^{2x} + 4Ce^{2x} + 2Ce^{2x} + 4Ee^{2x}$$

we want

$$y_p'' - 2y_p' - 3y_p = 4x - 5 + 6xe^{2x}$$

$\Leftrightarrow$

$$-3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}$$

$$\Leftrightarrow \begin{cases} -3A = 4 \\ -2A - 3B = -5 \\ -3C = 6 \\ 2C - 3E \end{cases} \Leftrightarrow \begin{cases} A = -\frac{4}{3} \\ B = \frac{23}{9} \\ C = -2 \\ E = -\frac{4}{3} \end{cases}.$$

$$\Rightarrow y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3. The general solution is

$$y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x} \quad \square$$

Example Solve $y'' - 3y' = 8e^{3x} + 4\sin x \quad (3)$

Step 1: Find y_c by solving

$$y'' - 3y' = 0$$

The auxiliary eq : $m^2 - 3m = 0$ has two roots

$m_1 = 0, \; m_2 = 3.$

$\Rightarrow \quad y_c = C_1 + C_2 e^{3x}$

Step 2: Find y_p.

$$(D-3)\, e^{3x} = 0, \quad (D^2+1)\sin x = 0$$

$\Rightarrow \quad L_1 = (D-3)(D^2+1)$

Applying L_1 to both side of the eq (3)

$$(D-3)(D^2+1)(D^2-3D)\, y = 0 \qquad (4)$$

The auxiliary eq of (4) is

$$(m-3)(m^2+1)(m^2-3m) = 0$$

Thus

$$y = \underbrace{C_1 + C_2 e^{3x}}_{y_c} + C_3 x e^{3x} + C_4 \cos x + C_5 \sin x$$

$\Rightarrow$

y_p has form $Ax e^{3x} + B\cos x + C\sin x$.

$$y_p' = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$$

$$y_p'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x} - B\cos x - C\sin x$$

Thus $\quad y_p'' - 3y_p' = 3Ae^{3x} + (-B-3C)\cos x + (3B - C)\sin x$

we want $y_p'' - 3y_p' = 8e^{3x} + 4\sin x$

$\Leftrightarrow$ $3Ae^{3x} + (-B-3C)\cos x + (3B-C)\sin x = 8e^{3x} + 4\sin x$

$\Leftrightarrow$ $3A = 8$, $-B-3C = 0$, $3B - C = 4$

$\Leftrightarrow$ $A = 8/3$, $B = 6/5$, $C = -2/5$.

$\Leftrightarrow$ $y_p = \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x$.

Step 3. The general solution of (3) is

$$y = C_1 + C_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x \quad \square$$

Example: Solve $y'' + y = x\cos x - \cos x$

Step 1: Find y_c:

$y'' + y = 0$ has the auxiliary eq $m^2 + 1 = 0$

$\Rightarrow$ $y_c = C_1 \cos x + C_2 \sin x$.

Step 2: Find y_p.

$(D^2+1)^2$ annihilates both $x\cos x$ and $\cos x$

$\Rightarrow$

$L_1 = (D^2+1)^2$.

Apply L_1 we get

$(D^2+1)^2(D^2+1)y = 0$

or $(D^2+1)^3 y = 0$

$$\Leftrightarrow \quad y = \underbrace{C_1\cos x + C_2 \sin x}_{y_c} + C_3 x\cos x + C_4 x \sin x + C_5 x^2 \cos x + C_6 x^2 \sin x$$

$\Rightarrow y_p = Ax\cos x + Bx\sin x + Cx^2\cos x + Ex^2\sin x.$

$\Rightarrow y_p'' + y_p = 4Ex\cos x - 4Cx\sin x + (2B+2C)\cos x$
$\qquad + (-2A+2E)\sin x$
$\qquad = x\cos x - \cos x$

$\Leftrightarrow \quad 4E = 1, \quad -4C = 0, \quad 2B+2C = -1, \quad -2A+2E = 0$

$\Leftrightarrow \quad A = \frac{1}{4}, \quad B = -\frac{1}{2}, \quad C = 0, \quad E = \frac{1}{4}$

$\Rightarrow \quad y_p = \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x.$

Step 3: The general solution is

$$y = c_1\cos x + c_2\sin x + \frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x.$$

$\square$

Example. Determine the form of a particular sol for

$$y'' - 2y' + y = 10e^{-2x}\cos x$$

Solution:

The complementary function is

$$y_c = c_1e^x + c_2xe^x.$$

The operation $L_1 = (D^2 + 4D + 5)$ annihilates $e^{-2x}\cos x$.

Applying the operator L_1 to our eq gives

$$(D^2+4D+5)(D^2-2D+1)y = 0$$

The roots of the auxiliary eq are $2-i, -2+i, 1, 1$

$\Rightarrow$

$$y = c_1e^x + c_2xe^x + c_3e^{-2x}\cos x + c_4e^{-2x}\sin x$$

$\Rightarrow \quad y_p = Ae^{-2x}\cos x + Be^{-2x}\sin x.$ $\square$

Example: Determine the form of a particular sol for

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}.$$

Solution:

$$D^3(5x^2 - 6x) = 0, \quad (D-2)^3 x^2 e^{2x} = 0, \text{ and}$$

$$(D-5)\, e^{5x} = 0.$$

Take $L_1 = D^3(D-2)^3(D-5)$. Apply L_1 to our eq gives

$$D^3(D-2)^3(D-5)(D^3 - 4D^2 + 4D)\, y = 0$$

or $\quad D^4(D-2)^5(D-5)\, y = 0.$

The roots of auxiliary eq are
$0, 0, 0, 0, 2, 2, 2, 2, 2, 5$

$\Rightarrow$

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) + (c_5 e^{2x} + c_6 x e^{2x} + c_7 x^2 e^{2x} + c_8 x^3 e^{2x} + c_9 x^4 e^{2x}) + c_{10} e^{5x}$$

Since $y_c = c_1 + c_5 e^{2x} + c_6 x e^{2x}$, we have

$\Rightarrow$

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2 e^{2x} + Fx^3 e^{2x} + Gx^4 e^{2x} + He^{5x}.$$

$\square$